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Groups generated by 3-state automata over a 2-letter alphabet, II

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Abstract

Classification of groups generated by 3-state automata over a 2-letter alphabet started in [BGK$^+$06] is continued.

1 Introduction

A comprehensive effort towards classification of all groups generated by invertible automata on three states acting on 2 letters ($(3, 2)$-automata) was started by our research group at Texas A&M University in [BGK$^+$06] and the current text presents further results in this direction.

The study of groups generated by finite automata is not new (it started in the beginning of 1960's). At the moment, the subject is going through a rather productive and mature phase in the sense that it is both contributing solutions to outstanding problems in various branches of mathematics (group theory, amenability, complex dynamics, etc.) and is developing and discovering its own fundamental concepts and theory leading to its own deep problems. A short account of the history of ideas related to the subject, as well as the most significant and interesting achievements of the theory can be found in [BGK$^+$06]. We also recommend the article [GNS00] and the book [Nek05] to readers interested in becoming more familiar with automaton groups.

Groups defined by automata on 2 states acting on two letters are already classified.

Theorem 1 ([GNS00]). *There are, up to isomorphism, 6 different groups generated by 2-state automata over a 2-letter alphabet. Namely, the trivial group, the cyclic group C_2 of order 2, Klein's Viergruppe $C_2 \times C_2$, the infinite cyclic group $\mathbb{Z}$, the infinite dihedral group D_∞ and the lamplighter group $\mathbb{Z} \wr C_2$.*

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We note that there are $2^3 \times 3^6 = 5832$ different (labeled) automata on three states acting on two letters. Obvious symmetries such as permutation of states, permutation of letters, and inversion of states, together with minimization, reduces the number of automata that needs to be checked to 194 (see [BGK+06] for more details). Through additional work we were able to reduce the number of non-isomorphic groups defined by these automata.

Theorem 2. *There are no more than* 124 *pairwise non-isomorphic groups defined by automata on 3 states over a 2-letter alphabet.*

For this text we chose 15 automata and we provide the basic information on each group defined by these automata. Section 5 contain a list of these 15 automata along with the basic information in a tabular form. Section 6 provides proofs of the claims in the tables as well as other information.

There are very few results that hold for the whole class of groups defined by automata on 3 states over a 2-letter alphabet. The reason is that this class contains many groups of rather distinct nature. The complete list contains finite groups, virtually free abelian groups, solvable groups (such as the lamplighter group $\mathbb{Z} \wr C_2$ and Baumslag-Solitar groups $BS(1, \pm 3)$), the free group F_3 of rank 3, the free product $C_2 * C_2 * C_2$, an amenable but not sub-exponentially amenable group (the Basilica group), and some examples about which we know very little.

Theorem 3. *There are* 6 *finite groups in the class:* $\{1\}$ *[1],* C_2 *[1090],* $C_2 \times C_2$ *[730],* D_4 *[847],* $C_2 \times C_2 \times C_2$ *[802] and* $D_4 \times C_2$ *[748].*

Theorem 4. *There are* 6 *abelian groups in the class:* $\{1\}$ *[1],* C_2 *[1090],* $C_2 \times C_2$ *[730],* $C_2 \times C_2 \times C_2$ *[802],* $\mathbb{Z}$ *[731] and* $\mathbb{Z}^2$ *[771].*

In addition, there are virtually free-abelian groups in the class (having $\mathbb{Z}$, $\mathbb{Z}^2$ [2212], $\mathbb{Z}^3$ [752] or $\mathbb{Z}^5$ [968] as subgroups of finite index).

Theorem 5. *The only free non-abelian group in the class is the free group of rank 3 generated by the Aleshin-Vorobets automaton [2240]. Moreover, the isomorphism class of this automaton group coincides with its equivalence class under symmetry.*

For the precise definition of symmetric automata see [BGK+06].

Theorem 6. *There are no infinite torsion groups in the class.*

2 Regular rooted tree automorphisms and self-similarity

Fix a finite alphabet $X = \{0, \ldots, d-1\}$ on d letters, $d \geq 2$. The set of all words X^* over X has the structure of a *regular rooted d-ary tree*, which we also denote X^*. The empty word is the *root* and each vertex w has d *children*, namely the

words wx, for $x \in X$. *Level* n of the tree X^* is the set X^n of all words of length n over X.

The *boundary* of the tree X^*, denoted X^ω, is the set of right infinite words over X^*. It corresponds to the set of infinite geodesic rays in X^* starting at the root.

Denote by $\mathrm{Aut}(X^*)$ the group of automorphisms of X^*. Note that any tree automorphism fixes the root and preserves the levels of the tree. Every automorphism f of X^* can be decomposed as

$$f = \alpha_f(f_0, \dots, f_{d-1}) \tag{1}$$

where $f_x \in \mathrm{Aut}(X^*)$, for $x \in X$, and α_f is a permutation of X. The action of f on X^* can be described in terms of the decomposition (1) as follows. Each of the d automorphisms f_x, $x \in X$, in the d-tuple $(f_0, \dots, f_{d-1})$ acts on the corresponding subtree xX^* consisting of the words over X starting in x, after which the permutation α_f permutes these subtrees. In other words

$$f(xw) = \alpha_f(x) f_x(w), \tag{2}$$

for a letter x in X and word w over X. The automorphism f_x, $x \in X$, are called the (first level) *sections* of f (and are sometimes denoted by $f|_x$), while α_f is called the *root permutation* of f. The notion of a section can be extended recursively as follows. The section of f at the root is f itself. For a word wx, where w is a word and x a letter over X, the section of f at wx is defined as the first level section $(f_w)_x$ of f_w at x.

The group $\mathrm{Aut}(X^*)$ decomposes algebraically as

$$\mathrm{Aut}(X^*) = \mathsf{Sym}(X) \ltimes (\mathrm{Aut}(X^*) \times \cdots \times \mathrm{Aut}(X^*)) = \mathsf{Sym}(X) \wr \mathrm{Aut}(X^*), \tag{3}$$

where $\wr$ is the *permutational wreath product* (the coordinates of $\mathrm{Aut}(X^*) \times \cdots \times \mathrm{Aut}(X^*) = \mathrm{Aut}(X^*)^X$ are permuted by $\mathsf{Sym}(X)$). The product of two automorphisms f and g of X^* is given by

$$\alpha_f(f_0, \dots, f_{d-1})\alpha_g(g_0, \dots, g_{d-1}) = \alpha_f\alpha_g(f_{g(0)}g_0, \dots, f_{g(d-1)}g_{d-1}).$$

A group G of tree automorphisms is *self-similar* if, for every g in G and a letter x in X there exists a letter y in X and an element h in G such that

$$g(xw) = yh(w),$$

for all words w over X. More simply put, a group G of tree automorphisms is self-similar if every section of every element in G is again an element of G.

The n-th *level stabilizer* $\mathsf{St}_G(n)$ of a group $G \leq \mathrm{Aut}(X^*)$ is the group of automorphisms in G that fix (pointwise) the vertices at level n (which implies that the vertices at the lower levels are also fixed).

A self-similar group G is called *self-replicating* if, for every vertex u, the homomorphism $\varphi_u : \mathsf{St}_G(u) \to G$ from the stabilizer u in G to G, given by $\varphi(g) = g_u$, is surjective.

A group G of tree automorphisms of X^* is *level transitive* if it acts transitively on each level of the tree X^*.

A group G of tree automorphisms is contracting if there exists a finite set $\mathcal{N} \subset G$, such that for every $g \in G$, there exists $N > 0$, such that $g_v \in \mathcal{N}$ for all vertices $v \in X^*$ of length at least N (see [Nek05]). The minimal set $\mathcal{N}$ with this property is called the *nucleus* of G. For a finitely generated group $G = \langle S \rangle$, where S is finite, the contracting condition is equivalent to the existence of constants κ, C, and N, with $0 \leq \kappa < 1$, such that $\ell(g_v) \leq \kappa \ell(g) + C$, for all vertices v of length at least N and $g \in G$ (the length $\ell(g)$ refers to the word length of the element g in G with respect to S).

A group G of tree automorphisms of X^* is a *regular weakly branch group* over its subgroup K if G acts spherically transitively on X^*, K is normal subgroup of G, and $K \times \cdots \times K$ is *geometrically contained* in K. The latter means that the first level stabilizer $\mathsf{St}_K(1)$ contains a subgroup of the form $K \times \cdots \times K$, where each factor acts on the corresponding subtree hanging below the first level of the tree. For more on (weakly) branch groups see [Gri00, BGŠ03].

3 Limit spaces and Schreier graphs

Let G be a finitely generated self-similar contracting group of tree automorphisms of X^*. Denote by $X^{-\omega}$ the space of left infinite words over X. Two elements $\ldots x_3 x_2 x_1, \ldots y_3 y_2 y_1 \in X^{-\omega}$ are *asymptotically equivalent* with respect to the action of G, if there exist a finite set $F \subset G$ and a sequence $\{f_k\}_{k=1}^{\infty}$ of elements in F such that

$$f_k(x_k x_{k-1} \ldots x_2 x_1) = y_k y_{k-1} \ldots y_2 y_1$$

for every $k \geq 1$.

The quotient space $\mathcal{J}_G$ of the topological space $X^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of the self-similar action of G.

The limit space $\mathcal{J}_G$ is metrizable and finite-dimensional (its dimension is bounded above by the size of the nucleus of G). If G acts spherically transitively, then the limit space $\mathcal{J}_G$ is connected.

We recall now the definition of Schreier graph. Let G be a group generated by a finite set S and let G acts on a set Y. The *Schreier graph* of the action (G, Y) is the graph $\Gamma(G, S, Y)$ whose set of vertices is Y and set of edges is $S \times Y$, where the initial vertex of the arrow (s, y) is y and its terminal vertex is $s(y)$. For $y \in Y$, the Schreier graph $\Gamma(G, S, y)$ of the action of G on the G-orbit of y is called the *orbital Schreier graph* of G at y.

Let $G = \langle S \rangle$, where S is finite, be a finitely generated subgroup of $\mathrm{Aut}(X^*)$ and consider the Schreier graphs $\Gamma_n(G, S) = \Gamma(G, S, X^n)$. Let $\xi = x_1 x_2 x_3 \ldots \in X^\omega$. Then the pointed Schreier graphs $(\Gamma_n(G, S), x_1 x_2 \ldots x_n)$ converge in the local topology to the pointed orbital Schreier graph $(\Gamma(G, S, \omega), \xi)$ (see [GŻ99] for more details on the topology of the space of pointed graphs).

Schreier graphs are related to the computation of the spectrum of the Markov operator M on the group. Given a finitely generated group $G = \langle S \rangle$, where S is

finite, acting on the tree X^*, there is a unitary representation of G in the space of bounded linear operators $\mathcal{H} = B(L_2(X^\omega))$, given by $\pi_g(f)(x) = f(g^{-1}x)$. The Markov operator M associated to this unitary representation is given by

$$M = \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \pi_s.$$

The spectrum of M for a self-similar group G can be approximated by the spectra of the finite dimensional Markov operators M_n, $n \geq 0$, related to the permutational representations of G provided by the action on the levels of the tree. The union of the spectra of M_n approximates the spectrum of M in the sense that

$$sp(M) = \overline{\bigcup_{n \geq 0} sp(M_n)}.$$

For more on this see [BG00].

4 Definition of automaton groups

We now formally describe the way finite automata define finitely generated self-similar groups of tree automorphisms.

A *finite invertible automaton* A is a quadruple $A = (Q, X, \tau, \rho)$ where Q is a finite set of *states*, X is a finite *alphabet* of cardinality $d \geq 2$, $\tau : Q \times X \to Q$ is a map, called *transition map*, and $\rho : Q \times X \to X$ is a map, called *output map*, such that, for each state q in Q, the restriction $\rho_q : X \to X$ given by $\rho_q(x) = \rho(q, x)$ is a permutation of X.

Each state q of the automaton A defines a tree automorphism of X^*, also denoted by q, by declaring that the root permutation of q is ρ_q and the section of q at x to be $\tau(q, x)$. Therefore

$$q(xw) = \rho_q(x)\tau(q, x)(w) \tag{4}$$

for all states q in Q, letters $x \in X$ and words w over X.

The group of tree automorphisms generated by the states of an invertible automaton $A = (Q, X, \tau, \rho)$ is called the *automaton group* defined by A and denoted by $G(A)$.

The boundary X^ω of the tree X^* is endowed with a natural metric (infinite words are close if they have long common beginnings). The group of isometries $\mathrm{Isom}(X^\omega)$ with respect to this metric is canonically isomorphic to $\mathrm{Aut}(X^*)$. Therefore the action of the automaton group $G(A)$ on X^* can be extended to an isometric action on X^ω using relation (4), which holds for infinite words w as well.

An invertible automaton A can be represented by a labeled directed graph, called Moore diagram, in which the vertices are the states of the automaton, each state q is labeled by its own root permutation ρ_q and, for each pair $(q, x) \in Q \times X$, there is an edge from q to $q_x = \tau(q, x)$ labeled by x. Moore diagrams of the 15 automata on 3 states that are subject of this article are provided in Section 5.

5 Selected groups

Information on selected groups generated by $(3,2)$-automata is provided in this section. The list appearing here supplements the list provided in [BGK$^+$06] and we keep the same notation.

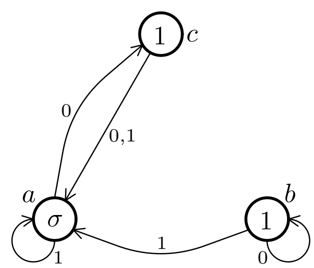
- Rels - a list of some relators in the group. In particular, lists of independent relators of length up to 20 are included. In many cases, the given relations are not sufficient (some of the groups are not finitely presented).
- SF - the size of the factors $G/\mathsf{St}_G(n)$, for $n \geq 0$.
- Gr - the values of the growth function $\gamma_G(n)$, for $n \geq 0$, with respect to the generating system consisting of a, b, and c.

In addition, in each case, a histogram for the spectral density of the operator M_9 corresponding to the action on level 9 of the tree is provided for every automaton. Approximation of the limit space is provided for the automaton 775.

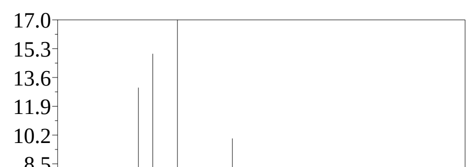
The relations (and some other data) were obtained using the computer algebra system GAP [GAP06] and software developed by Y. Muntyan and D. Savchuk.

Automaton number 741

$a = \sigma(c,a)$ Group:
$b = (b,a)$ Contracting: *no*
$c = (a,a)$ Self-replicating: *yes*



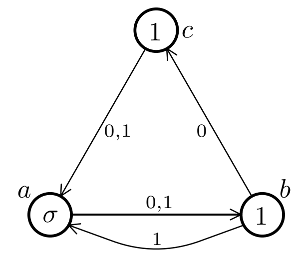
Rels: ca^2, $b^{-1}a^{-1}cb^{-1}ababa$,
$bab^{-1}ca^{-1}b^{-1}aba$, $a^{-1}b^{-1}a^{-1}b^{-1}cabc^{-1}ab$,
$a^{-1}b^{-1}a^{-1}b^{-1}acbc^{-1}ab$, $b^{-1}c^{-3}b^{-1}cbcbc$,
$a^{-1}b^{-1}abc^{-1}a^{-1}bab^{-1}c$, $a^{-1}b^{-1}aba^{-1}c^{-1}bab^{-1}c$.
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,7,29,115,441,1643

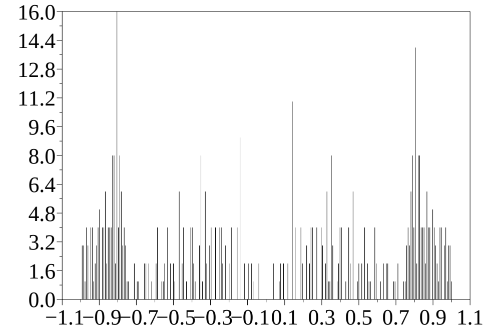


Automaton number 752

$a = \sigma(b, b)$ Group: Virtually $\mathbb{Z}^3$
$b = (c, a)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*

Rels: a^2, b^2, c^2, $acbabacbab$, $acacbacacb$, $abcabcacbacb$, $acbcbabacbcbab$, $abcbacbabcbacb$, $acbcacbacbcacb$, $acacbcbacacbcb$, $abcbcabcacbcbacb$, $acbcbcbabacbcbcbab$, $abcbcbacbabcbcbacb$.
SF: 2^0,2^1,2^3,2^5,2^7,2^8,2^{10},2^{11},2^{13}
Gr: 1,4,10,22,46,84,140,217,319,448

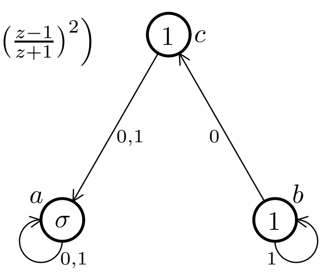


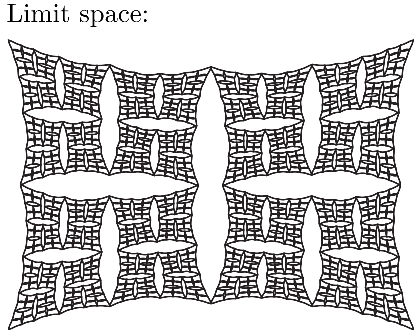


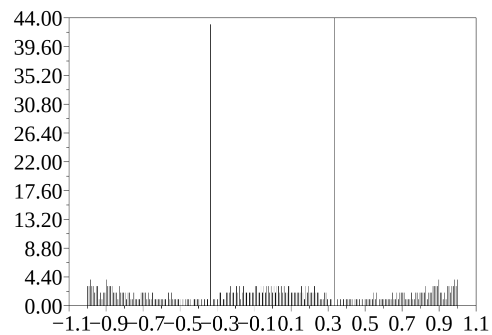
Automata number 775 **and** 783

$a = \sigma(a, a)$ $a = \sigma(c, c)$ Group: $C_2 \ltimes IMG\left(\left(\frac{z-1}{z+1}\right)^2\right)$
$b = (c, b)$ 783: $b = (c, b)$ Contracting: *yes*
$c = (a, a)$ $c = (a, a)$ Self-replicating: *yes*

Rels: a^2, b^2, c^2, $acac$, $acbcbabcbcabcbabcb$, $acbcbabcbacbcbabcb$, $abcbacbcbcacbcbcabcb$, $acbcbacbcbabcbcabcbc$, $acbcbacbcbcacbcbcabcbc$
SF: 2^0,2^1,2^2,2^4,2^6,2^9,2^{15},2^{26},2^{48}
Gr: 1,4,9,17,30,51,85,140,229,367,579
Limit space:

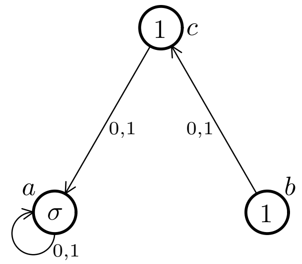


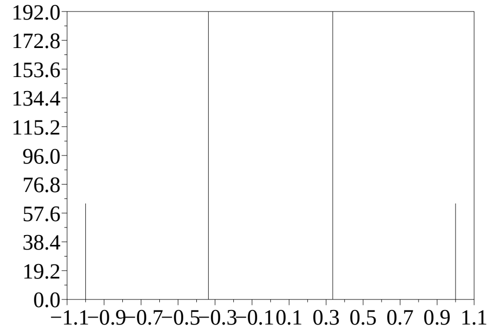




Automaton number 802

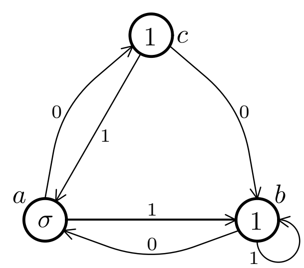
$a = \sigma(a, a)$ Group: $C_2 \times C_2 \times C_2$
$b = (c, c)$ Contracting: *yes*
$c = (a, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2, $abab$, $acac$, $bcbc$
SF: 2^0,2^1,2^2,2^3,2^3,2^3,2^3,2^3,2^3
Gr: 1,4,7,8,8,8,8,8,8,8,8

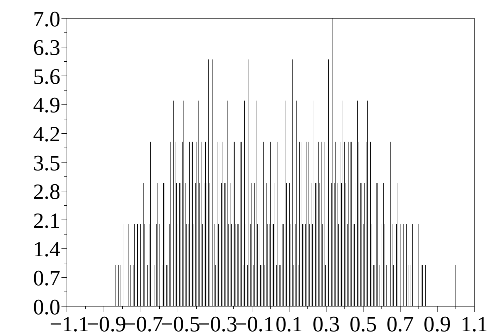




Automaton number 843

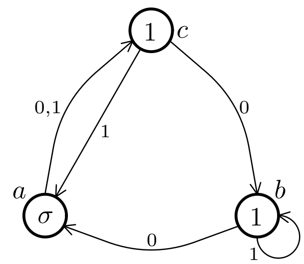
$a = \sigma(c, b)$ Group:
$b = (a, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $acab^{-1}a^{-2}cab^{-1}aba^{-1}c^{-1}ba^{-1}c^{-1}$,
$acab^{-1}a^{-2}cb^{-1}ab^{-1}caba^{-1}c^{-2}ba^{-1}bc^{-1}$,
$acb^{-1}ab^{-1}ca^{-2}cab^{-1}ac^{-1}ba^{-1}bc^{-1}ba^{-1}c^{-1}$,
$acb^{-1}ab^{-1}ca^{-2}cb^{-1}ab^{-1}cac^{-1}ba^{-1}bc^{-2}ba^{-1}bc^{-1}$
SF: 2^0,2^1,2^3,2^5,2^8,2^{14},2^{24},2^{43},2^{81}
Gr: 1,7,37,187,937,4687

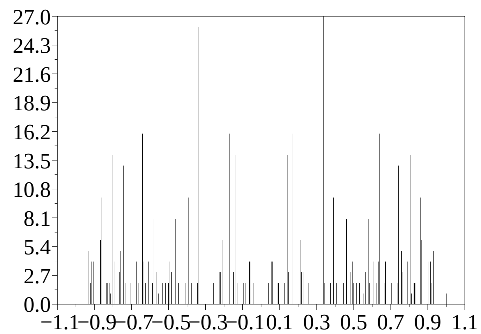




Automaton number 846

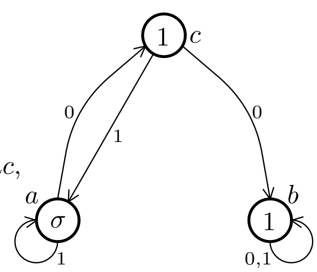
$a = \sigma(c, c)$ Group: $C_2 * C_2 * C_2$
$b = (a, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *no*
Rels: a^2, b^2, c^2
SF: 2^0,2^1,2^3,2^5,2^7,2^{10},2^{13},2^{16},2^{19}
Gr: 1,4,10,22,46,94,190,382,766,1534

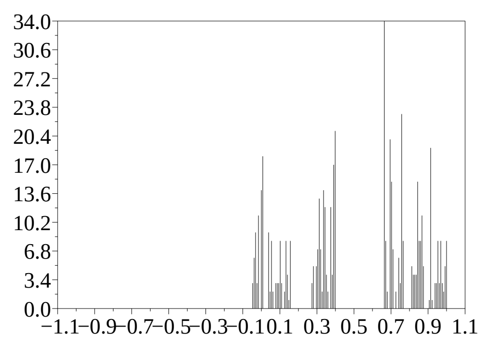




Automaton number 849

$a = \sigma(c, a)$ Group:
$b = (b, b)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: b, $a^{-1}c^{-1}ac^{-1}a^{-1}cac$, $a^{-1}c^{-2}ac^{-1}a^{-1}c^2ac$, $a^{-1}c^{-1}ac^{-2}a^{-1}cac^2$, $a^{-4}ca^2c^{-2}a^2c$, $a^{-1}c^{-3}ac^{-1}a^{-1}c^3ac$, $a^{-1}c^{-2}ac^{-2}a^{-1}c^2ac^2$, $a^{-1}c^{-1}ac^{-3}a^{-1}cac^3$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}
Gr: 1,5,17,53,153,421,1125,2945,7589





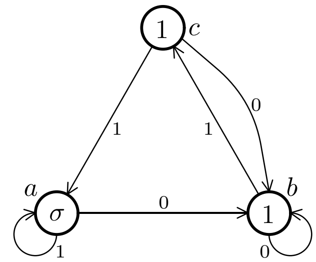
Automaton number 875

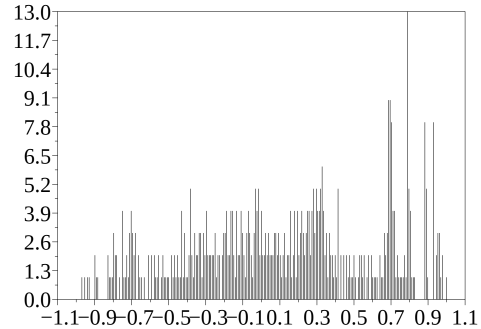
$a = \sigma(b, a)$ Group:
$b = (b, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*

Rels: $a^{-1}ca^{-1}c$, $b^{-1}cb^{-1}c$, $a^{-1}ba^{-1}ba^{-1}ba^{-1}b$, $a^{-1}ba^{-1}bc^{-1}ac^{-1}ba^{-1}b$

SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}

Gr: 1,7,33,143,607,2563





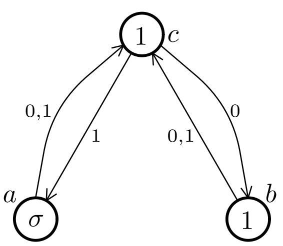
Automaton number 891

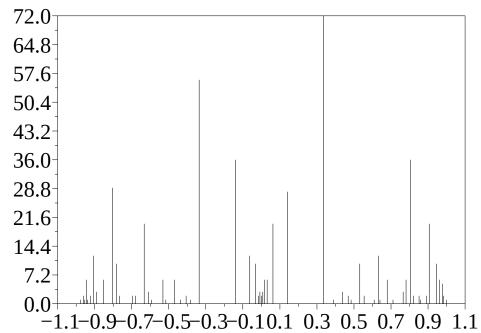
$a = \sigma(c, c)$ Group: $C_2 \ltimes$ *Lampighter*
$b = (c, c)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*

Rels: a^2, b^2, c^2, *abab*, *acabcbacbacb*, *acabcabcbacb*, *acacabcacbacacbacacb*, *acacabcacabcacbacacb*, *acacabcbcbacbcbacacb*, *acacabcbcabcbcbacacb*, *acabcbcabcacbacbcbac*

SF: 2^0,2^1,2^3,2^6,2^7,2^9,2^{10},2^{11},2^{12}

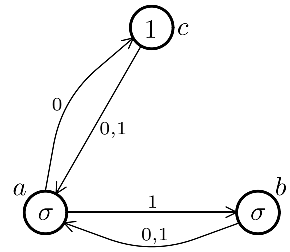
Gr: 1,4,9,17,30,51,82,128,198,304,456



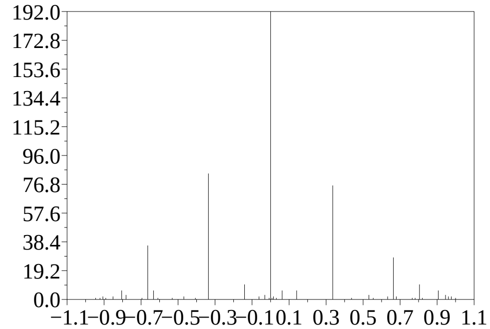


Automaton number 2193

$a = \sigma(c, b)$ Group: contains *Lamplighter* group
$b = \sigma(a, a)$ Contracting: *no*
$c = (a, a)$ Self-replicating: *yes*

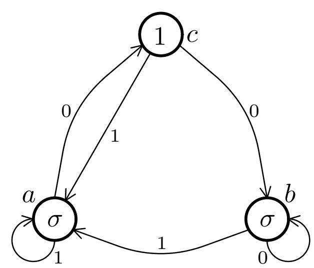


Rels: $b^{-1}c^{-1}bc$, $b^{-2}c^2$, $b^{-1}cb^{-1}c$, a^{-4}, b^{-4}, $b^{-1}c^{-2}b^{-1}$, $b^{-1}c^{-1}b^{-1}c^{-1}$, $a^{-2}ba^2b$, $a^{-1}b^{-1}c^{-1}a^{-1}cb$, $a^{-2}ba^{-2}b$, $a^{-2}ca^2c$, $a^{-2}ca^{-2}c$, $b^{-1}c^{-1}a^{-1}bca^{-1}$, $a^{-1}b^{-1}a^{-2}b^{-1}a^{-1}$.
SF: 2^0,2^1,2^3,2^6,2^7,2^9,2^{10},2^{11},2^{12}
Gr: 1,7,27,65,120,204,328,512,792,1216

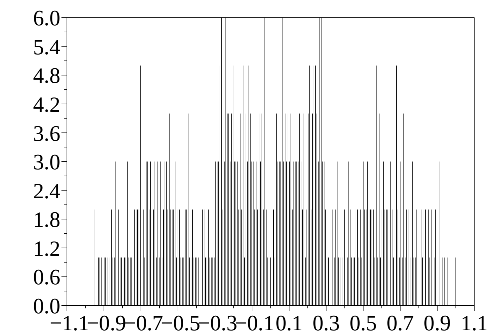


Automaton number 2280

$a = \sigma(c, a)$ Group:
$b = \sigma(b, a)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*

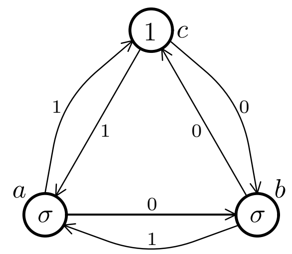


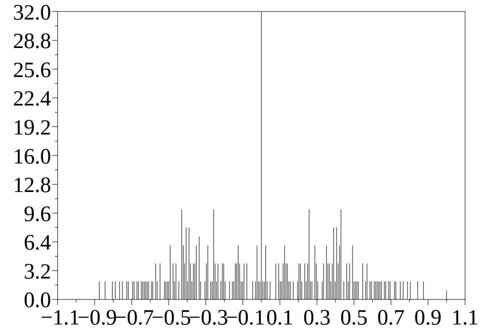
Rels: $a^{-1}ba^{-1}b$, $b^{-1}cb^{-1}c$, $b^{-1}ca^{-1}ba^{-1}c$, $a^{-2}b^2a^{-1}b^{-1}ab$, $a^{-2}bab^{-2}ab$, $b^{-2}c^2b^{-1}c^{-1}bc$, $b^{-2}cbc^{-2}bc$, $a^{-1}ca^{-1}ca^{-1}ca^{-1}c$, $b^{-1}ca^{-1}cb^{-1}ca^{-1}c$, $a^{-1}bab^{-1}a^{-1}b^2a^{-1}$, $b^{-1}cbc^{-1}b^{-1}c^2b^{-1}$, $a^{-2}bac^{-1}bc^{-1}b^{-1}ab$.
SF: 2^0,2^1,2^3,2^7,2^{13},2^{25},2^{47},2^{90},2^{176}
Gr: 1,7,33,143,597,2465



Automaton number 2294

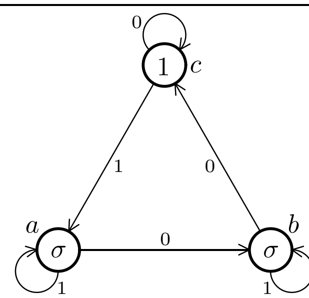
$a = \sigma(b, c)$ Group: *Baumslag-Solitar group* $B(1, -3)$
$b = \sigma(c, a)$ Contracting: *no*
$c = (b, a)$ Self-replicating: *yes*
Rels: $b^{-1}ca^{-1}c$, $(ca^{-1})^a(ca^{-1})^3$
SF: 2^0,2^1,2^2,2^4,2^6,2^8,2^{10},2^{12},2^{14}
Gr: 1,7,33,127,433,1415

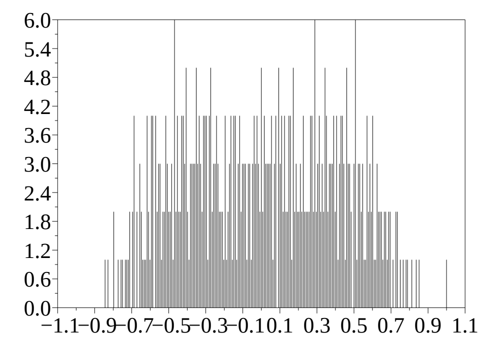




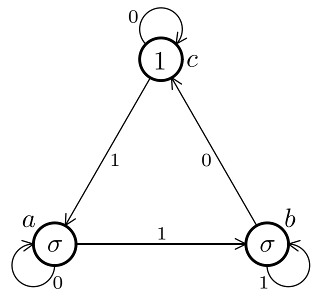
Automaton number 2396

$a = \sigma(b, a)$ Group: *Boltenkov group*
$b = \sigma(c, b)$ Contracting: *no*
$c = (c, a)$ Self-replicating: *yes*
Rels: $acb^{-1}ca^{-2}cb^{-1}cac^{-1}bc^{-2}bc^{-1}$,
$acb^{-1}ca^{-2}cb^{-1}a^2c^{-1}b^{-1}a^2c^{-1}bc^{-1}a^{-1}bca^{-2}bc^{-1}$,
$acb^{-1}a^2c^{-1}b^{-1}a^{-1}cb^{-1}cbca^{-2}bc^{-2}bc^{-1}$,
$bcb^{-1}ca^{-1}b^{-1}cb^{-1}a^2c^{-1}ac^{-1}ba^{-2}bc^{-1}$
SF: 2^0,2^1,2^3,2^6,2^{12},2^{24},2^{46},2^{90},2^{176}
Gr: 1,7,37,187,937,4687





Automaton number 2398

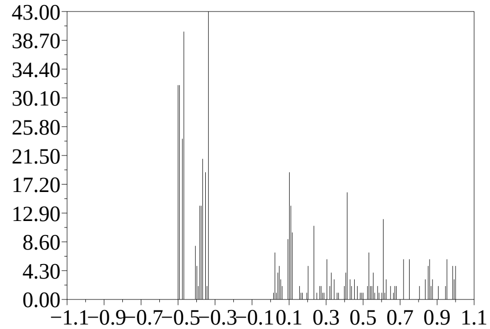


$a = \sigma(a, b)$	Group: *Dahmani Group*
$b = \sigma(c, b)$	Contracting: *no*
$c = (c, a)$	Self-replicating: *yes*

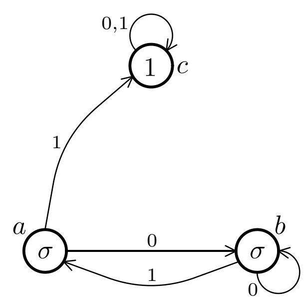
Rels: cba, $b^{-1}a^{-1}b^2ca^2$, $a^{-2}c^{-1}acb^{-1}ab$, $a^{-1}c^{-1}b^{-1}acaba^{-1}$, $b^{-1}a^2b^{-1}a^{-1}b^2a^{-1}$, $c^{-1}b^{-3}c^{-1}ba^{-2}$, $c^{-1}b^{-1}a^{-1}b^2cab^{-1}$, $c^{-1}ab^{-1}a^{-1}b^2cb^{-1}$, $a^{-3}c^{-1}b^{-2}ac^{-1}$, $ca^{-1}bc^{-1}a^{-1}ca^3$, $cabcba^{-1}c^{-1}b^{-1}a$, $ab^{-3}c^{-1}ba^{-2}b$, $ba^{-1}c^{-1}acb^{-1}abc$, $bc^{-1}b^{-1}acaba^{-1}c$

SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,7,31,127,483,1823



Automaton number 2852

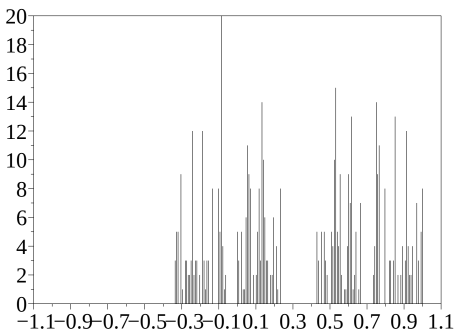


$a = \sigma(b, c)$	Group: Isomorphic to G_{849}
$b = \sigma(b, a)$	Contracting: *no*
$c = (c, c)$	Self-replicating: *yes*

Rels: c, $a^{-2}bab^{-2}ab$, $a^{-2}ba^{-1}bab^{-2}ab^{-1}ab$, $a^{-1}bab^{-1}ab^{-2}aba^{-1}ba^{-1}$

SF: 2^0,2^1,2^3,2^6,2^{12},2^{23},2^{45},2^{88},2^{174}

Gr: 1,5,17,53,153,429,1189



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6 Proofs of some facts about the selected groups

741: For G_{741}, we have $a = \sigma(c, a)$, $b = (b, a)$ and $c = (a, a)$.

The states a and c form a 2-state automaton generating $\mathbb{Z}$ (see Theorem 1 and its proof in [GNS00]). Since $b^n = (b^n, a^n)$ we see that b also has infinite order and that G_{741} is not contracting (b^n would have to belong to the nucleus for all n).

752: For G_{752}, we have $a = \sigma(b, b)$, $b = (c, a)$ and $c = (a, a)$.

We claim that this group contains $\mathbb{Z}^3$ as a subgroup of index 4. It is contracting with nucleus consisting of 41 elements.

Since $a^2 = (b^2, b^2)$, $b^2 = (c^2, a^2)$ and $c^2 = (a^2, a^2)$, all generators have order 2.

Let $x = ca$, $y = babc$, and consider the subgroup $K = \langle x, y\rangle$. Direct computations show that x and y commute ($xy = ((cbab)^{ca}, abcb) = ((y^{-1})^x, abcb)$ and $yx = (cbab, abcb) = (y^{-1}, abcb)$). Conjugating by $\gamma = (\gamma, bc\gamma)$ leads to the self-similar copy K' of K generated by $x' = \sigma((y')^{-1}, (x')^{-1})$ and $y' = \sigma((y')^{-1}x', 1)$, where $x' = x^\gamma$ and $y' = y^\gamma$. Since $(x')^2 = ((x')^{-1}(y')^{-1}, (y')^{-1}(x')^{-1})$ and $(y')^2 = ((y')^{-1}x', (y')^{-1}x')$, the virtual endomorphism of K' is given by the matrix

$$A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

The eigenvalues $\lambda = -\frac{1}{2} \pm \frac{1}{2}i$ of this matrix are not algebraic integers, hence, according to [NS04], the group K' is free abelian of rank 2, and so is K.

Since all generators have order 2, the subgroup $H = \langle ba, cb\rangle$ has index 2 in G_{752}. The stabilizer $\mathsf{St}_H(1)$ of the first level has index 2 in H and is generated by cb, cb^{ba} and $(ba)^2$. Conjugating these three generators by $g = (1, b)$, we obtain

$$\begin{aligned} g_1 &= (cb)^g &&= (x^{-1}, 1), \\ g_2 &= \left((cb)^{ba}\right)^g &&= (1, \quad x), \\ g_3 &= \left((ba)^2\right)^g &&= (y^{-1}, y). \end{aligned}$$

Therefore, g_1, g_2, and g_3 commute. If $g_1^{n_1} g_2^{n_2} g_3^{n_3} = 1$, then all sections must be trivial, implying that $x^{-n_1}y^{-n_3} = x^{n_2}y^{n_3} = 1$. But K is free abelian, whence $n_i = 0$, $i = 1, 2, 3$. Thus, $\mathsf{St}_H(1)$ is a free abelian group of rank 3.

775: For G_{775}, we have $a = \sigma(a, a)$, $b = (c, b)$ and $c = (a, a)$.

This group was mentioned in the previous paper [BGK$^+$06], where it was proved that it is conjugate to G_{783} and isomorphic to G_{2205}. Some additional properties of this group were mentioned without a proof. Here we list and prove some of those properties.

Conjugating the generators by $g = \sigma(g, g)$ we obtain

$$a' = \sigma(a', a'), \qquad b' = (b', c'), \qquad c' = (a', a'),$$

where $a' = a^g$, $b' = b^g$ and $c' = c^g$. The tree automorphisms a', b' and c' are precisely the generators of the group G_{793} and we choose to work with this group

instead of G_{775}. In fact, from now on the generators of G_{793} will be denoted by a, b and c and the group G_{793} will be denoted by G. Thus

$$a = \sigma(a, a), \qquad b = (b, c), \qquad c = (a, a)$$

and $G = \langle a, b, c\rangle$.

Since $a^2 = (a^2, a^2)$, $b^2 = (b^2, c^2)$ and $c^2 = (a^2, a^2)$, all three generators have order 2. Moreover, since $ac = \sigma$ has order 2, a and c commute, i.e., $\langle a, c\rangle = C_2 \times C_2$ is the Klein Viergruppe. Denote this subgroup of G by A.

We note that the element $x = ba$ has infinite order. Indeed,

$$x = ba = (b, c)\sigma(a, a) = \sigma(ca, ba) = \sigma(\sigma, x).$$

Therefore $x^2 = (x\sigma, \sigma x) = (x, \sigma, \sigma, x)$. Since x^2 fixes 00, and has x as a section, x must have infinite order (the element x cannot have odd order since it acts nontrivially at the root; on the other hand $x^{2n} = 1$ implies $x^n = 1$, so x cannot have even order as well).

Proposition 1. *The subgroup $H = \langle x, y\rangle$ of G, where $x = ba$ and $y = cabc$ is torsion free. Moreover*

$$G = A \ltimes H = (C_2 \times C_2) \ltimes H$$

Proof. The equalities

$$\begin{aligned}
x^a &= abaa = ab = x^{-1} & y^a &= acabca = cbac = y^{-1} \\
x^b &= bbab = ab = x^{-1} & y^b &= bcabcb = bacbacab = xy^{-1}x^{-1} \\
x^c &= cbac = y^{-1} & y^c &= ccabcc = ab = x^{-1}
\end{aligned}$$

show that H is normal in G (in fact they even show that H is the normal closure of x in G). Since $\{x, y, a, c\}$ is a generating set for G it follows that $G = AH$. The fact that $A \cap H = 1$ and therefore $G = A \ltimes H$ will follow from the fact that H is torsion free.

The first level decompositions of $x^{\pm 1}$ and $y^{\pm 1}$ and the second level decompositions of x and y are

$$\begin{aligned}
&x = \sigma(\sigma, x) \\
&y = cabc = \sigma aaba\sigma = \sigma ba\sigma = x^\sigma = \sigma(x, \sigma) \\
&x^{-1} = \sigma(x^{-1}, \sigma) \\
&y^{-1} = \sigma(\sigma, x^{-1}) \\
&x = \sigma(\sigma(1, 1), \sigma(\sigma, x)) = \mu(1, 1, \sigma, x) \\
&y = x^\sigma = \mu(\sigma, x, 1, 1),
\end{aligned}$$

where $\mu = \sigma(\sigma, \sigma)$ permutes the first two levels of the tree as $00 \leftrightarrow 11$, $10 \leftrightarrow 01$, which we encode as the permutation $\mu = (03)(12)$. For a word w over $\{x^{\pm 1}, \sigma\}$ let $\#_x(w)$ and $\#_\sigma(w)$ denote the total number of appearances of x

and x^{-1} and the number of appearances of σ in w, respectively. Note that a tree automorphisms represented by a word w over $\{x^{\pm 1}, \sigma\}$ cannot be trivial unless both $\#_x(w)$ and $\#_\sigma(w)$ are even (we use this claim freely in the further considerations). This is because x and x^{-1} act as the permutation $(03)(12)$ on the second level, σ acts as $(02)(13)$, these permutations have order 2, commute, and their product is $(01)(23)$, which is not trivial.

Let g be an element of H that can be written as $g = z_1 z_2 \dots z_n$, for some $z_i \in \{x^{\pm 1}, y^{\pm 1}\}$, $i = 1, \dots, n$.

We claim that if n is odd, the element g cannot have order 2. Assume otherwise. For z in $\{x^{\pm 1}, y^{\pm 1}\}$ denote $z' = \sigma z$. Thus $x' = (\sigma, x)$, $y' = (x, \sigma)$ and so on. We have

$$g^2 = (z_1 z_2 \dots z_n)^2 = (z_1')^\sigma z_2' (z_3')^\sigma z_4' \dots (z_n')^\sigma z_1' (z_2')^\sigma \dots z_n' = (w_0, w_1),$$

where the words w_i over $\{x^{\pm 1}, \sigma\}$ have the property that

$$\#_x(w_i) = \#_\sigma(w_i) = n, \tag{5}$$

for $i = 1, 2$. This is because exactly one of z_i' and $(z_i')^\sigma$ contributes $x^{\pm 1}$ to w_0 and σ to w_1, respectively, while the other contributes the same letters to w_1 and w_0, respectively. But now g^2 cannot be 1, since (5) shows that neither w_0 nor w_1 can be 1.

Assume H is not torsion free. Then, since an automorphism of a binary tree can only have 2-torsion, there exists an element of order 2 in H and let g be such an element of the shortest possible length over $\{x, y\}$. Let this length be n and let $g = z_1 z_2 \dots z_n$, for some $z_i \in \{x^{\pm 1}, y^{\pm 1}\}$, $i = 1, \dots, n$.

Since n must be even, we have

$$g = z_1 z_2 \dots z_n = (z_1')^\sigma z_2' \dots (z_{n-1}')^\sigma z_n' = (w_0, w_1),$$

where w_0 and w_1 are words over $\{x^{\pm 1}, \sigma\}$ whose orders in H divide 2 and the order of at least one of them is 2. We have

$$\#_x(w_0) \equiv \#_\sigma(w_0) \equiv \#_x(w_1) \equiv \#_\sigma(w_0) \mod 2. \tag{6}$$

Indeed, $\#_x(w_i) \equiv \#_\sigma(w_i) \mod 2$, because $\#_x(w_i) + \#_\sigma(w_i) = n$ is even. On the other hand, whenever z_i' or $(z_i')^\sigma$ contributes $x^{\pm 1}$ or σ to w_0, respectively, it also contributes σ or $x^{\pm 1}$ to w_1, respectively. Thus $\#_x(w_0) = \#_\sigma(w_1)$ and $\#_\sigma(w_0) = \#_x(w_1)$.

If all the numbers in (6) are even we have that w_0 and w_1 represent elements in H (due to the fact that $x^\sigma = y$) and can be rewritten as words over $\{x^{\pm 1}, y^{\pm 1}\}$ of lengths at most $\#_x(w_0) = n - \#_\sigma(w_0)$ and $\#_x(w_1) = n - \#_\sigma(w_1)$, respectively. Either both of these lengths are shorter than n and therefore none of them can represent an element of order 2 in H or one of the words w_i is a power of x and the other is trivial. However, x has infinite order, thus again showing that $g = (w_0, w_1)$ cannot have order 2.

Finally, assume that all the numbers in (6) are odd. Then, for $i = 1, 2$, w_i can be rewritten as σu_i, where u_i are words over $\{x^{\pm 1}, y^{\pm 1}\}$ of odd length.

Let $w_0 = \sigma u_0 = \sigma t_1 \dots t_m$, where m is odd, and t_j are letters in $\{x^{\pm 1}, y^{\pm 1}\}$, $j = 1, \dots, m$. Then

$$w_0 = t'_1 (t'_2)^\sigma \dots (t'_{m-1})^\sigma t'_m = (w_{00}, w_{01}),$$

where w_{00} and w_{01} are words of odd length m over $\{x^{\pm 1}, \sigma\}$, exactly one of which has even number of σ's. That word can be rewritten as a word over $\{x^{\pm 1}, y^{\pm 1}\}$ of odd length. However, the element in H represented by such a word cannot have order 2, as proved above. $\square$

Lemma 1. *The group H is not metabelian.*

Proof. Assume that H is metabelian. Consider $[x, y] = x^{-1}y^{-1}xy = (x^2, x^{-2})$ and $[x, y^{-1}] = x^{-1}yxy^{-1} = (y^2, x^{-2})$. We have

$$\begin{aligned}
[[x, y], [x, y^{-1}]] = 1 \quad & \Rightarrow \\
[(x^2, x^{-2}), (y^2, x^{-2})] = 1 \quad & \Rightarrow \text{(consider the 0 coordinate)} \\
[x^2, y^2] = 1 \quad & \Rightarrow \\
[(x\sigma, \sigma x), (\sigma x, x\sigma)] = 1 \quad & \Rightarrow \text{(consider the 0 coordinate)} \\
[x\sigma, \sigma x] = 1 \quad & \Rightarrow \\
[(x, \sigma), (\sigma, x)] = 1 \quad & \Rightarrow \text{(consider the 0 coordinate)} \\
[x, \sigma] = 1. \quad &
\end{aligned}$$

The last commuting relation then implies that $x = x^\sigma = y$, which in turn implies that $x = \sigma$, which is impossible since x has infinite order. $\square$

Lemma 2. *The group H'' geometrically contains $H'' \times H'' \times H'' \times H''$, i.e.,*

$$H'' \times H'' \times H'' \times H'' \preceq H''.$$

Proof. The equalities

$$x^2 = (x, \sigma, \sigma, x)$$
$$y^{-1}x^2 y = (y, x^{-1}\sigma x, \sigma, x)$$

imply that

$$H'' \times \langle \sigma, x^{-1}\sigma x \rangle'' \times \langle \sigma \rangle'' \times \langle x \rangle'' \preceq H''.$$

Since $\langle \sigma, x^{-1}\sigma x \rangle$ is dihedral (both generators have order 2) and $\langle \sigma \rangle$ and $\langle x \rangle$ are cyclic, we have

$$H'' \times 1 \times 1 \times 1 \preceq H''.$$

It follows that $x^{-1}(H'' \times 1 \times 1 \times 1)x = 1 \times 1 \times 1 \times H'' \preceq H''$.

Similarly, the equalities

$$y^2 = (\sigma, x, x, \sigma)$$
$$x^{-1}y^2 x = (\sigma, x, y, x^{-1}\sigma x)$$

imply that $1 \times 1 \times H'' \times 1 \preceq H''$ and then $y^{-1}(1 \times 1 \times H'' \times 1)y = 1 \times H'' \times 1 \times 1 \preceq H''$. $\square$

Corollary 1. *The group G is a weakly branch group.*

Proof. The group G, being an infinite self-similar group acting on a binary three, acts spherically transitively (see Proposition 2 in [BGK$^+$06]). The group H'' is normal in G, since it is characteristic in the normal subgroup H. Lemma 1 implies that H'' is not trivial and Lemma 2 then implies that G_{783} is a weakly branch group. □

802: $C_2 \times C_2 \times C_2$. For G_{802} we have $a = \sigma(a, a)$, $b = (c, c)$, and $c = (a, a)$.

Straightforward calculations give that the group is abelian and has the indicated structure (there are 8 elements in all and the nontrivial ones have order 2).

A quick way to see that the group G_{802} is finite is to first note that the automaton consisting of the states a and c generates the Klein Viergruppe $C_2 \times C_2$ (see Theorem 1 and its proof in [GNS00]) and all sections of b at first level belong to this automaton. More generally, the following proposition holds (see, for example, [BS06b]).

Proposition 2. *Let G be a self-similar group of tree automorphisms of X^*, let S be a finite generating set for G, and let F be a finite set of automorphisms from* $\operatorname{Aut} X^*$*, such that there exists a level k in the tree with the property that all sections of all automorphisms in F at this level belong to G (in particular, F could be a set of finitary automorphisms). Then*

$$\gamma_G(n) \precsim \gamma_{\langle G,F\rangle}(n) \precsim \left(\gamma_G(n)\right)^{|X|^k}.$$

where $\gamma_G(n)$ and $\gamma_{\langle G,F\rangle}(n)$ are the growth functions of the groups G and $\langle G, F\rangle$ with respect to the generating sets S and $S \cup F$ respectively.

843: For G_{843}, we have $a = \sigma(c, b)$, $b = (a, b)$, and $c = (b, a)$.

The element $c^{-1}a = \sigma(a^{-1}c, 1)$ acts spherically transitively on the tree (it is conjugate to the adding machine) and therefore has infinite order. Also, $c^{-1}a$ has infinite order because $(c^{-1}a)^2 = (a^{-1}c, a^{-1}c)$. Since $(c^{-1}ab^{-1}a)^2$ fixes the vertex 000 and its section at this vertex equals to $c^{-1}a$, we get that $c^{-1}ab^{-1}a$ also has infinite order. But the element $c^{-1}ab^{-1}a$ has itself as a section at vertex 10. Hence, G_{843} is not contracting.

846: $C_2 * C_2 * C_2$. For G_{846} we have $a = \sigma(c, c)$, $b = (a, b)$, and $c = (b, a)$.

The automaton 846 is sometimes called Bellaterra automaton, because it was investigated during the Advanced Course on Automata Groups, held in Bellaterra, Spain in 2004. A proof was given in [Nek05], and here we present the original proof, which uses action of the automaton dual to the automaton 846.

A *dual* automaton is the automaton that is obtained after swapping the alphabet and the set of states, as well as the transition and output functions, in a given automaton. Namely, if $\mathcal{A} = (Q, X, \tau, \rho)$, then its dual automaton is $\mathcal{A}^* = (X, Q, \overline{\rho}, \overline{\tau})$, where $\overline{\rho}(x, q) = \rho(q, x)$, $\overline{\tau}(x, q) = \tau(q, x)$. The dual automaton represents transitions in the automata $\mathcal{A}^n$, $n \geq 0$, when they consume input

letters from X: the image of the action of $x \in X$ on a word $q_1 q_2 \dots q_n \in Q^n$ is the state $q_1' q_2' \dots q_n'$ of $\mathcal{A}^n$ reached after processing the input letter x starting from the initial state $q_1 q_2 \dots q_n$. In particular, the orbits of the semigroup generated by the automaton $\mathcal{A}^*$ represent the states reachable from a given state in the automata $\mathcal{A}^n$, $n \geq 0$.

Notion of dual automaton appeared in [Ale83] where it was used to prove that the group generated by two initial automata is free (unfortunately the proof was not complete, and a complete proof can be found in [VV06]). A particularly interesting case is when both $\mathcal{A}^*$ and $(\mathcal{A}^{-1})^*$ are invertible, such an automaton $\mathcal{A}$ is called *bi-reversible*. Examples of such automata are automaton 2240 generating a free group with three generators [VV05], automaton 846 (under consideration), and various automata constructed in [GM05], generating free groups of various ranks.

If the automaton $\mathcal{A}^*$ is invertible then its action on the set Q^* is in fact a group action, which coincides with the action of the tree automorphisms group generated by $\mathcal{A}^*$ and we can apply existing techniques to study the action of $\mathcal{A}^*$ on sets Q^n.

Consider the group G_{846}. Its generators a, b, and c have order 2, therefore to prove that G_{846} is isomorphic to $C_2 * C_2 * C_2$ we need to show that no word in $w \in W_n$, $n \geq 1$, is trivial in G_{846}, where $W_n \subset \{a, b, c\}^*$ is the set of all words of length n which do not contain squares of the letters a, b, and c. For any $n > 0$, the set of words in W_n that are not trivial in G_{846} is nonempty. For instance, such a word is $w = abcbcb\cdots$ (this is because b and c act trivially on the level 1 of the tree, while a permutes the two vertices at level 1). If the state w in the n-th power of the automaton 846 is reachable from some state $w' \in W_n$ then the element w' is not trivial in G_{846}. Therefore it is sufficient to show that the dual automaton acts transitively on the sets W_n, $n \geq 1$.

The dual automaton to 846 is the invertible automaton defined by

$$\begin{array}{rcl} A & = & (acb)(B, A, A), \\ B & = & (ac)(A, B, B). \end{array} \tag{7}$$

The group $D = \langle A, B \rangle$ does not act spherically transitively on the ternary tree over the alphabet $\{a, b, c\}$ (for instance the orbit of a^2 under action of D consists only of a^2, b^2, and c^2). Let us prove that D acts transitively on the sets W_n. The set $T' = \bigcup_{n \geq 0} W_n$ of all words that do not contain squares of the letters in $\{a, b, c\}$ is a subtree of the ternary tree $\{a, b, c\}^*$ and this subtree is invariant under the action of D. The root of T' is connected to the vertices a, b and c, which, in turn, are roots of 3 binary trees. The action of D on this subtree can be derived from the relations (7). The generators A and B act as follows:

$$\begin{array}{rcl} A & = & (acb)(B_a, A_b, A_c) \\ B & = & (ac)(A_a, B_b, B_c) \end{array} \tag{8}$$

where $A_a, A_b, A_c, B_a, B_b, B_c$ are the automorphisms of the binary trees hanging down from the vertices a, b and c. All these binary trees can be naturally

identified with the tree $\{0,1\}^*$, where the action of $A_a, A_b, \ldots, B_c$ is defined by

$$\begin{array}{rcl} A_a &=& (A_b, A_c), \\ A_b &=& \sigma(B_a, A_c), \\ A_c &=& \sigma(B_a, A_b), \\ B_a &=& \sigma(B_b, B_c), \\ B_b &=& \sigma(A_a, B_c), \\ B_c &=& \sigma(A_a, B_b). \end{array} \tag{9}$$

The algorithm deciding the spherical transitivity of an element acting on the binary tree (see Algorithm 1 in [BGK$^+$06]) shows that the element B_b acts level transitively. Thus D acts transitively on the levels of T'. Indeed, D acts transitively on the first level, B stabilizes vertex b and its section at b is B_b.

849: For G_{849} we have $a = \sigma(c, a)$, $c = (1, a)$ and $b = 1$.

The element $a^2c = (ac, ca^2)$ is nontrivial because its section at vertex 0 is ac, which acts nontrivially on the first level. The section of $(a^2c)^2$ at vertex 00 coincides with a^2c, hence this element has infinite order. On the other hand, the section of a^2c at vertex 100 coincides with a^2c, which shows that G_{849} is not contracting.

The group G_{849} is regular weakly branch over G'_{849} because it is self-replicating and $[a^{-1}, c] \cdot [c, a] = ([a, c], 1)$.

Conjugating the generators of G_{849} by the automorphism $\mu = \sigma(\mu, c^{-1}\mu)$ yields the recursion

$$x = \sigma(yx, 1), \qquad y = (x, 1),$$

where $x = a^\mu$ and $y = c^\mu$. Since

$$x = \sigma(yx, 1), \qquad yx = \sigma(yx, x),$$

and the last recursion defines the automaton 2852, we see that $G_{849} \cong G_{2852}$. See G_{2852} for further information.

875. For G_{875} we have $a = \sigma(b, a)$, $b = (b, c)$, and $c = (b, a)$.

Let us prove that a has infinite order. This will be accomplished by proving that the orbit of $110^\infty \in X^\omega$ under the action of a is infinite.

For every $w \in X^*$, $a(w0^\infty) = a(w0)a|_{w0}(0^\infty) = a(w0)b(0^\infty) = a(w0)0^\infty$. Therefore all points in the orbit of 110^∞ under the action of a end in 0^∞. For a word w ending in 0^∞ define the *head* of w to be shortest word u such that $w = u0^\infty$. The length of the head of any infinite word ending in 0^∞ cannot decrease under the action of a. Indeed, for every $w \in X^*$,

$$a(w010^\infty) = a(w0)a|_{w0}(10^\infty) = a(w0)b(10^\infty) = a(w0)10^\infty \quad \text{and}$$
$$a(w110^\infty) = a(w11)a|_{w11}(0^\infty) = a(w11)a(0^\infty) = a(w11)10^\infty.$$

On the other hand, the length of the head along the orbit cannot stabilize, because in this case the orbit would be finite and we must have $a^k(110^\infty) = 110^\infty$, for some $k \geq 1$. But this is impossible since $a(110^\infty) = 0010^\infty$ and the

length of the head does not decrease. Thus the orbit is infinite and a has infinite order.

Since $c = (b, a)$ and $b = (b, c)$, the elements b and c also have infinite order. Finally, since $b = (b, c)$ and b has infinite order, G_{875} is not contracting.

891: $C_2 \ltimes$ *Lamplighter*. For G_{891} we have $a = \sigma(c, c)$, $b = (c, c)$, and $c = (b, a)$. All generators have order 2.

Consider the subgroup $H = \langle x, y \rangle$, where $x = ac$ and $y = cb$. We will prove that H is isomorphic to the Lamplighter group $\mathbb{Z} \ltimes C_2$. We have

$$x = ac = \sigma(cb, ca) = \sigma(y, x^{-1}), \qquad y = cb = (bc, ac) = (y^{-1}, x).$$

Note that $xy = \sigma$.

Consider the elements $s_n = \sigma^{y^n} = y^{-n}xy^{n+1}$, $n \in \mathbb{Z}$. We have, for $n > 0$, $s_0 s_1 \cdots s_{n-1} = x^n y^n$ and $s_{-n}s_{-n+1} \cdots s_{-1} = y^n x^n$. Further, $s_n = y^{-n}\sigma y^n = \sigma(x^{-n}y^{-n}, y^n x^n)$ and $s_{-n} = y^n \sigma y^{-n} = \sigma(x^n y^n, y^{-n}x^{-n})$. Hence,

$$s_n = \sigma(s_{-1}s_{-2} \cdots s_{-n}, s_{-n} \cdots s_{-2}s_{-1})$$

and

$$s_{-n} = \sigma(s_0 s_1 \cdots s_{n-1}, s_{n-1} \cdots s_1 s_0).$$

Now by induction we get that the depth of s_n is $2n + 1$ for $n \geq 0$ and the depth of s_{-n} is $2n$ for $n > 0$ (by *depth* of a finitary element we mean the lowest level at which all sections of the element are trivial). Therefore, all s_i are different, commute (because for each i and each level m all sections of s_i at level m are equal), and have order 2 (they are all conjugate to σ). Note that this proves that y has infinite order and that $H = \langle x, y \rangle = \langle y, \sigma \rangle \cong \mathbb{Z} \ltimes C_2$.

The subgroup H has index at most 2 in G_{891} because the generators of G_{891} are of order 2. In fact, it has index exactly 2, since there is no relation of odd length in G_{891}. Indeed, let w be a word over $\{a, b, c\}$ representing the identity in G_{891}. Then $\#_a(w)$ must be even (since a is the only generator that acts nontrivially on the first level). Further, $\#_c(w)$ must also be even. This is because w can be decomposed as $w = (w_0, w_1)$ where w_0 and w_1 are words over $\{a, b, c\}$ such that $\#_a(w_0) + \#_a(w_1) = \#_c(w)$ (only the generator c contributes a to the next level in the decomposition). Further, $\#_c(w_0) = \#_c(w_1) = \#_a(w) + \#_b(w)$, since both a and b contribute c to both coordinates on the next level. Thus if $\#_b(w)$ were odd (and we already know that $\#_a(w)$ is even) then none of the words w_0 and w_1 would represent the identity. Note that we proved by this that every word w representing the identity in G_{891} must have even number of occurrences of a b and c, which shows not only that c does not belong to H, but also that the abelianization of G_{891} is $C_2 \times C_2 \times C_2$ (and the commutator consists precisely of the elements represented by words w for which $\#_a(w)$, $\#_b(w)$, and $\#_c(w)$ are even).

Therefore, the group $G_{891} = \langle c, H \rangle$ has the structure of a semidirect product $G_{891} = C_2 \ltimes (\mathbb{Z} \wr C_2)$ and the action of c on H is given by $(x)^c = (ac)^c = ca = x^{-1}$ and $y^c = (cb)^c = bc = y^{-1}$. It follows that G_{891} is solvable group of exponential growth.

Since y has infinite order, stabilizes the vertex 00 and has itself as a section at this vertex, we conclude that G_{891} is not contracting.

2193. For G_{2193} we have $a = \sigma(c, b)$, $b = \sigma(a, a)$, and $c = (a, a)$.

For $x = ca^{-1}$ and $y = ab^{-1}$, we have $x = \sigma(ab^{-1}, ac^{-1}) = \sigma(y, x^{-1})$ and $y = (ba^{-1}, ca^{-1}) = (y^{-1}, x)$. It is proved above (see G_{891}), that $\langle x, y\rangle$ is not contracting and is isomorphic to the Lamplighter group. Thus G_{2193} is neither torsion, nor contracting and has exponential growth.

2280. For G_{2280} we have $a = \sigma(c, a)$, $b = \sigma(b, a)$, and $c = (b, a)$.

Let us prove that a has infinite order. We will prove that the orbit of 10^∞ under iterations of a^2 is infinite. Recall the definition of a head for a word over $\{0, 1\}$ ending in 0^∞ (see G_{875}). For every word $w \in X^*$, we have $a^2(w10^\infty) = a^2(w) \cdot a^2|_w(10^\infty)$. The automorphism a^2 has 6 distinct sections defined by

$$\begin{aligned} a^2 &= (ac, ca), & ac &= \sigma(cb, a^2), & ca &= \sigma(ac, ba) \\ cb &= \sigma(ab, ba), & ba &= (ac, ba), & ab &= (ab, ca). \end{aligned}$$

Since

$$\begin{aligned} &a^2(10^\infty) = ab(10^\infty) = 1110^\infty, \\ &ac(10^\infty) = ca(10^\infty) = cb(10^\infty) = 0010^\infty, \text{ and} \\ &ba(10^\infty) = 10110^\infty \end{aligned}$$

we see that the length of the head increases by 3 at each application of a^2 and a has infinite order.

Since $a^2 = (ac, ca)$, the element ca also has infinite order. The element ab has infinite order, because $ab = (ab, ca)$. Therefore, G_{2280} is not contracting.

2294: Baumslag-Solitar group $BS(1, -3)$. For G_{2294} we have $a = \sigma(b, c)$, $b = \sigma(c, a)$, and $c = (b, a)$.

The automaton satisfies the conditions of Proposition 1 in [BGK+06], which implies that cb has infinite order. Since $a^2 = (cb, bc)$, the generator a also has infinite order.

Let $\mu = ca^{-1}$. We have $\mu = ca^{-1} = \sigma(ac^{-1}, 1) = \sigma(\mu^{-1}, 1)$, which shows that μ has infinite order (it acts spherically transitively on the binary tree). Since $bc^{-1} = \sigma(cb^{-1}, 1) = \sigma((bc^{-1})^{-1}, 1)$, we see that $bc^{-1} = \mu = ca^{-1}$ and therefore $G_{2294} = \langle \mu, a\rangle$.

We claim that $a\mu a^{-1} = \mu^{-3}$ in G_{2294}. We have

$$a\mu a^{-1} = aca^{-2} = \sigma(bc^{-1}, cac^{-1}b^{-1}), \qquad \mu^{-3} = \sigma(\mu, \mu^2).$$

Therefore it is sufficient to show that $cac^{-1}b^{-1} = ca^{-1}ca^{-1}$, i.e., $ac^{-1}b^{-1}ac^{-1}a = 1$. This is clear since $ac^{-1}b^{-1}ac^{-1}a = (ca^{-2}ca^{-1}b, 1)$ and $ca^{-2}ca^{-1}b$ is conjugate of the inverse of $ac^{-1}b^{-1}ac^{-1}a$. This completes the proof.

For realizations of $BS(1, m)$ for any value of m, $m \neq \pm 1$, see [BŠ06a].

2396: *Boltenkov group.* For G_{2396} we have $a = \sigma(b, a)$, $b = \sigma(c, b)$, and $c = (c, a)$.

This group was studied by Boltenkov. He proved that it is torsion free and that the monoid generated by a, b, and c is free. Here we provide his proofs and show that the group is not contracting.

Proposition 3. *The monoid generated by a, b, and c is free.*

Proof. Suppose there are some relations. Let $w = u$ be a relation for which $\max(|w|, |u|)$ minimal. Assume that none of the words w and u is empty. Clearly, w and u end in different letters (since cancelation holds). Then $w = \sigma_w(w_0, w_1) = \sigma_u(u_0, u_1) = u$, where σ_w, σ_u are permutations in $\{1, \sigma\}$. It follows that $w_0 = u_0$ and $w_1 = u_1$ are also relations. Consider different cases.

Let w end in b and u end in c. Then w_0 and u_0 both end in c. This means, by minimality, that $w_0 = u_0$ as words. Thus, in particular, $|u| = |w|$. Since $b \neq c$ in G_{2396} (their actions differ at level 1) the length of w and u is at least 2. We can unambiguously recover the second to last letter in w and u. Indeed, the second to last letter in u_0 can be only b or c (these are the only possible sections at 0 of the three generators), while the second to last letter of w_0 can be only a or b (these are the only possible sections at 1 of the three generators). Thus $w_0 = u_0 = \dots bc$, $w = \dots bb$, and $u = \dots ac$. As $bb \neq ac$ in G_{2396} (their actions differ at level 1), the length of w and u is at least 3. Iterating the procedure, we obtain that $w_0 = u_0 = b \dots bbc$, and therefore $w = \dots ababb$, $u = \dots babac$. As $|u| = |w|$, the actions of w and u differ at level 1 and we get a contradiction.

Let w end in a and u end in b or c. Then u_0, w_0 end in b and c, respectively, and the situation is reduced to the case above.

Therefore, there are no nontrivial relations in G_{2396} of the form $w = u$, for nonempty words w and u over $\{a, b, c\}$.

If u is an empty word, then $w_0 = 1 = w_1$, which shows that $w_0 = w_1$ is also a minimal relation, which is impossible because both w_0 and w_1 are nonempty. □

For a group word w over $\{a, b, c\}$, define the exponent $\exp_a(w)$ of a in w to be the sum of the exponents of all the occurrences of a and a^{-1} in w (define $\exp_b(w)$ and $\exp_c(w)$ analogously). Let $\exp(w) = \exp_a(w) + \exp_b(w) + \exp_c(w)$. Note that if $w = \sigma_w(w_0, w_1)$, then $\exp(w) = \exp(w_0) = \exp(w_1)$.

Lemma 3. *If $w = 1$ in G_{2396} then* $\exp(w) = 0$

Proof. Suppose not. Choose a freely reduced group word w over $\{a, b, c\}$ such that $w = 1$ in G_{2396}, $\exp(w) \neq 0$, and w has minimal length among all such words. Let $w = (w_0, w_1)$. Then w_0, w_1 also represent 1 in G_{2396} and $\exp(w_0) = \exp(w_1) = \exp(w) \neq 0$. Therefore w_0 and w_1 are not empty and, by minimality, their length must be equal to the length of w. This implies that w does not contain ac^{-1}, bc^{-1}, ca^{-1}, or cb^{-1} as a subword (this is because $ac^{-1} = \sigma(bc^{-1}, 1)$ and $bc^{-1} = \sigma(1, ba^{-1})$).

By the same argument w_0 and w_1 must not have the above 4 words as subwords as well, implying that w does not have $ab^{-1} = (ab^{-1}, bc^{-1})$ or its inverse ba^{-1} as a subword. Thus, since w is reduced, we have that $1 = w =$

$W_1(a^{-1}, b^{-1}, c^{-1})W_2(a, b, c)$, and we obtain a relation between positive words over $\{a, b, c\}$, contradicting Proposition 3. $\square$

Note that if $w = 1$ in G_{2396} then $\exp_a(w)$, $\exp_b(w)$ and $\exp_c(w)$ are even. Indeed, clearly $\exp_a(w) + \exp_b(w)$ must be even (by looking at action on level 1), which implies by the previous lemma that $\exp_c(w)$ is even. Further, if $w = (w_0, w_1)$, then also $\exp_a(w_0)+\exp_b(w_0)$ and $\exp_a(w_1)+\exp_b(w_1)$ are even. We have that $\exp_a(w) + \exp_b(w) = \exp_b(w_0) + \exp_b(w_1)$, $\exp_a(w) + \exp_c(w) = \exp_a(w_0)+\exp_a(w_1)$. Therefore $2\exp_a(w)+\exp_b(w)+\exp_c(w)$ is even, implying that $\exp_b(w)$ is even. Since both $\exp_b(w)$ and $\exp_c(w)$ are even, so must be $\exp_a(w)$.

Proposition 4. *The group G_{2396} is torsion free.*

Proof. Suppose not. Then G_{2396} has an element w of order 2. By considering the sections we can also suppose that w does not belong to the stabilizer of the first level, i.e. $w = \sigma(w_0, w_1)$. It follows that $w^2 = (w_1w_0, w_0w_1) = 1$. We have that, modulo 2, $\exp_b(w_0w_1) = \exp_b(w_0)+\exp_b(w_1) = \exp_a(w)+\exp_b(w)$. Since $\exp_b(w_0w_1)$ is even, we conclude that $\exp_a(w) + \exp_b(w)$ is even, contradicting the assumption that w does not stabilize the first level. $\square$

The element $b^{-1}a = (c^{-1}b, b^{-1}a)$ is nontrivial (the a exponent is odd) and therefore has infinite order by Proposition 4. On the other hand, $b^{-1}a$ fixes vertex 1 and its section at this vertex is $b^{-1}a$, implying that G_{2396} is not contracting.

2398: This group was studied by Dahmani in [Dah05]. It is self-replicating, not contracting, weakly regular branch group over its commutator subgroup.

2852=G_{849}. For G_{2852} we have $a = \sigma(b, 1)$, $b = \sigma(b, a)$, $c = 1$.

Since $a^2 = (b, b)$ and $ab = (b, ba)$, the group G_{2852} is self-replicating and spherically transitive.

The group G_{2852} is a regular weakly branch group over G'_{2852}. Indeed, $[a^{-1}, b]\cdot[b, a] = ([a, b], 1)$. Spherical transitivity and the self-replicating property then imply that $G'_{2852} \times G'_{2852} \preceq G'_{2852}$. Since G_{2852} is not abelian ($[b, a] = (b^{-1}ab, a^{-1}) \neq 1$, because $a \neq 1$), G'_{2852} is nontrivial and G_{2852} is a regular weakly branch over G'_{2852}.

We have $b^2 = (ab, ba)$, $ba = (ab, b)$, and $ab = (b, ba)$. Thus b^2 fixes the vertex 00 and has b as a section at that vertex. Since b is nontrivial, this implies that b has infinite order, and therefore so does ab. On the other hand, ab fixes vertex 10 and has itself as a section at that vertex, implying that G_{2852} is not contracting.

We claim that the monoid generated by a and b is free (and therefore the group has exponential growth).

Indeed, let w be a nonempty word in $\{a, b\}^*$. If $w = 1$ in G_{2852}, then w contains both a and b, because they both have infinite order (a has infinite order since $a^2 = (b, b)$). Suppose the length of w is minimal among all nonempty words over $\{a, b\}$ representing the identity element in G_{2851}. Then one of the

sections of w will be shorter than w (since $a|_1$ is trivial), nonempty (since $b|_0$ and $b|_1$ are nontrivial), and will represent the identity in G_{2851}, which contradicts the minimality assumption. Thus $w \neq 1$ in G_{2851}, for any nonempty word in $\{a,b\}^*$.

Now take any nonempty words $w, v \in \{a,b\}^*$ with minimal sum $|w| + |v|$ such that $w = v$ in G_{2852}. Let $w = \sigma_w(w_0, w_1)$ and $u = \sigma_u(u_0, u_1)$, where $\sigma_w, \sigma_w \in \{1, \sigma\}$. Suppose w ends in a and v ends in b. Then $w_1 = v_1$ in G. If $w = a$ then $w_1 = 1$ in G and the word v_1 is nontrivial, which is impossible. If w contains more than one letter then w_1 ends in b, v_1 ends in a, and $|w_1| < |w|$ and $|v_1| \leq |v|$. Thus we have a shorter relation $w_1 = v_1$, contradicting the minimality assumption.

See G_{849} for the isomorphism between G_{2852} and G_{849}.

Conjugating the generators of G_{2852} by the automorphism $\mu = \sigma(b\mu, \mu)$ yields the recursion

$$x = \sigma(y, 1), \qquad y = \sigma(xy, 1),$$

where $x = a^\mu$ and $y = b^\mu$. Since

$$y = \sigma(xy, 1), \qquad xy = (xy, y),$$

and the last recursion defines the automaton 933, we see that $G_{2852} \cong G_{933}$.

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